Properties of a Set of Symmetric Maxwellian Equations

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A set of fully symmetric maxwellian equations is proposed, with a quantum mechanical coupling term between matter and fields. Its relativistic transformation properties and conservation laws are presented. Dual, monopole-like solutions are described, which have properties consistent with those of the Dirac electron and magnetic monopole. The spatial extent of the monopole fields is proposed to be bound within two extreme radial limits r_p and R_0 such that $\alpha \ln (R_0/r_p) \equiv 1$, where $\alpha \simeq 1/137$ is the electromagnetic fine structure constant, yielding for the ratio R_0/r_p a very large number in the order of the ratio of the so-called universe radius to the Planck length.

Key words: Dirac Electron, Magnetic Monopole, Maxwell's Equations

Ample evidence has been given for an isomorphism between Maxwell's equations and the Dirac equation [1, 2]. In other words, it is possible to transform the Dirac equation into a set of equations whose form is strongly reminiscent of Maxwell's equations. This has led us to investigate a new set of maxwellian equations which, despite analogies with those derived directly from the Dirac equation [1, 2], differs from these in the source terms. The difference appears essential, as is suggested by the properties of the proposed set. The latter is thus presented here, with a brief description of some of its properties and monopole-like solutions. Dipole-like solutions will be given elsewhere in an extended version of this article.

The proposed set reads

$$\nabla \cdot \boldsymbol{E} = -\boldsymbol{k}_0 \cdot \boldsymbol{B}, \tag{1a}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{k}_0 \times \boldsymbol{B}, \tag{1b}$$

$$\nabla \cdot \boldsymbol{B} = -\boldsymbol{k}_0 \cdot \boldsymbol{E}, \tag{1c}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{k}_0 \times \boldsymbol{E}, \tag{1d}$$

where $\boldsymbol{k}_0$ is a radial vector at the point P (r, θ, φ) where the fields are calculated (Fig. 1), and of magnitude $k_0 = m_0 c/\hbar$, the inverse of the Compton wavelength for a particle of mass m_0. This choice for the magnitude of $\boldsymbol{k}_0$ is suggested by the above-mentioned relationship with the Dirac equation.

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Equations (1) can be shown to be invariant in form (i.e. covariant) under the familiar Lorentz transformations for the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, providing the following relations hold between fixed and moving frames (assuming a uniform motion along the z-direction of a Cartesian coordinates system):

$$k'_{0x} = k_{0x};\ k'_{0y} = k_{0y};\ k'_{0z} = k_{0z}/\sqrt{1-\beta^2}, \tag{2a}$$

$$D_{s'}\, u' = (D_s\, u')\frac{\partial s}{\partial s'} + \frac{\partial u'}{\partial t}\frac{\partial t}{\partial s'},$$
$$\frac{\partial u'}{\partial t'} = \frac{\partial u'}{\partial t}\frac{\partial t}{\partial t'} + (D_z\, u')\frac{\partial z}{\partial t'}, \tag{2b}$$

where $\beta = v/c$, s is x, y, or z and $D_s \equiv (\partial/\partial s \pm k_{0s})$, $u' = (E' \pm B')$, $\partial z/\partial t' = v/\sqrt{1-\beta^2}$, $E'_z = E_z$, $E'_x = (E_x - \beta B_y)/\sqrt{1-\beta^2}$, etc.

The transformation (2a) reflects the Lorentz contraction of the Compton wavelength in the direction of motion. The transformations (2b) generalize the well-known rules [3] when these are applied to functions $u_1 = u \exp(\pm \boldsymbol{k}_{0s} \cdot (\boldsymbol{s} - \boldsymbol{s}_p))$, in the limit $\boldsymbol{s} \to \boldsymbol{s}_p$, where $\boldsymbol{s}_p$ is the coordinate vector to the point of observation P. It is indeed easy to verify that in this limit one has $\lim_{\boldsymbol{s}\to\boldsymbol{s}_p} u_1 = u$, and $\lim_{\boldsymbol{s}\to\boldsymbol{s}_p} \partial u_1/\partial s = D_s\, u$.

The requirement for invariance thus provides the prescription on how to account for quantum uncertainty when performing a frame transformation.

Proceeding now to the conservation laws [4], one first takes the divergence of (1b) and (1d), and uses (1a) and (1c) with the rules of vector calculus. One then finds that here the continuity relations reduce to mere identities, so that continuity is automatically implied in the proposed set of equations.

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Similarly, one finds the so-called energy conservation law in the form

$$c\nabla\cdot[2(\boldsymbol{E}\times\boldsymbol{B})/c]=\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{E}\cdot\boldsymbol{E}+\boldsymbol{B}\cdot\boldsymbol{B}), \quad (3)$$

where one notes in particular the vanishing of all source-field interaction terms, consistent with the continuity equations being mere identities.

Finally, one also obtains the momentum conservation law in the form

$$\nabla\cdot(2\mathbf{S})=\frac{\partial}{\partial t}(2(\boldsymbol{E}\times\boldsymbol{B})/c)-2(2\boldsymbol{k}_0(\boldsymbol{E}\cdot\boldsymbol{B})), \quad (4)$$

where $\mathbf{S}$ is the Maxwell stress tensor [4].

It is interesting that the last term of (4) can be transformed in such a way that the conservation law takes the same form as Eq. (15) on p. 99 of [4], exhibiting the same type of Lorentz forces acting on the sources.

Based on the above conservation law, we then define the local density of linear particle momentum as

$$\boldsymbol{p}=2(\boldsymbol{E}\times\boldsymbol{B})/c-2\boldsymbol{k}_0\int_0^t 2(\boldsymbol{E}\cdot\boldsymbol{B})\,\mathrm{d}t. \quad (5)$$

Since the vector product $\boldsymbol{k}_0\times\boldsymbol{p}$ is also conserved, we use (5) to define the local density of intrinsic angular momentum accordingly as

$$\boldsymbol{m}=(\boldsymbol{r}_0/2)\times(2(\boldsymbol{E}\times\boldsymbol{B})/c), \quad (6)$$

where $\boldsymbol{r}_0=(h/m_0c)\,\boldsymbol{k}_0/|\boldsymbol{k}_0|$.

It will be seen below that this expression for $\boldsymbol{m}$ yields a spin of $\pm\hbar/2$ for the monopole-like solutions to (1), whereas (5) yields the momentum m_0c of the Dirac electron [5]. The spin can then be interpreted according to (6) as arising from the particle's "Zitterbewegung" with amplitude $r_0/2=\hbar/2m_0c$ and momentum m_0c.

It will be shown elsewhere (using the dipole-like solutions to (1)), that (5) also yields the relativistically correct value, $p=\sqrt{E^2-(m_0c^2)^2}/c$, for the particle momentum, without the classical and incorrect factor 4/3 [6], thanks to the matter momentum contribution from the last term in (5).

Now, from the point of view of electromagnetic theory, the "Zitterbewegung", and thus also the spin, of an electric (magnetic) charge must give rise to a magnetic (electric) field. That this is indeed the case is described next, with the monopole solutions to the set (1).

Using spherical coordinates and assuming azimuthal symmetry, i.e. $\partial/\partial\varphi\equiv 0$, one can verify that (1) admit of two independent and dual solutions, namely an electric and a magnetic monopole, respectively.

The electric monopole has fields

$$E_r=(q/4\pi r^2)\sin\omega t$$

$$B_\theta=(k_0q/4\pi r)\left(\frac{\cos\theta\pm1}{\sin\theta}\right)\sin\omega t$$

$$B_\varphi=-(\omega q/4\pi c r)\left(\frac{\cos\theta\pm1}{\sin\theta}\right)\cos\omega t \quad (7)$$

with $\omega=\pm\omega_0=\pm m_0c^2/\hbar$.

The magnetic monopole has fields

$$B_r=(g/4\pi r^2)\sin\omega t,$$

$$E_\theta=(k_0g/4\pi r)\left(\frac{\cos\theta\pm1}{\sin\theta}\right)\sin\omega t,$$

$$E_\varphi=+(\omega g/4\pi c r)\left(\frac{\cos\theta\pm1}{\sin\theta}\right)\cos\omega t, \quad (8)$$

with $\omega=\pm\omega_0=\pm m_0c^2/\hbar$.

The symbols $q=e$ and g stand for the electric and magnetic charges, respectively. It will be shown elsewhere that the Dirac quantization condition for monopoles, $qg/4\pi\hbar c=1/2$ [7], can also be derived from the above solutions.

As an illustration of the above derivations, we now compute the linear and angular momentum of the electric monopole. Using (5), (6), (7), and a volume element $\mathrm{d}\tau=r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$, one finds

(a) for the linear momentum along the $\boldsymbol{\Omega}$-axis (Fig. 1)

$$\boldsymbol{P}_\Omega=\frac{1}{c}\int 2(\boldsymbol{E}\times\boldsymbol{B})_\theta\sin\theta\,\mathrm{d}\tau$$
$$=\pm[\alpha\ln(R_0/r_\mathrm{p})](m_0c)\sin 2\omega_0t; \quad (9)$$

(b) for the time averaged angular momentum along the same $\boldsymbol{\Omega}$-axis

$$\langle\boldsymbol{M}_\Omega\rangle=-\frac{1}{c}\int\langle r_0E_rB_\theta\rangle\sin\theta\,\mathrm{d}\tau$$
$$=\mp[\alpha\ln(R_0/r_\mathrm{p})]\,\hbar/2, \quad (10)$$

where $\alpha\equiv q^2/4\pi\hbar c$ is the electromagnetic fine structure constant, and r_p and R_0 are hypothetical lower and upper radial bounds on the domain of the fields (Figure 1).

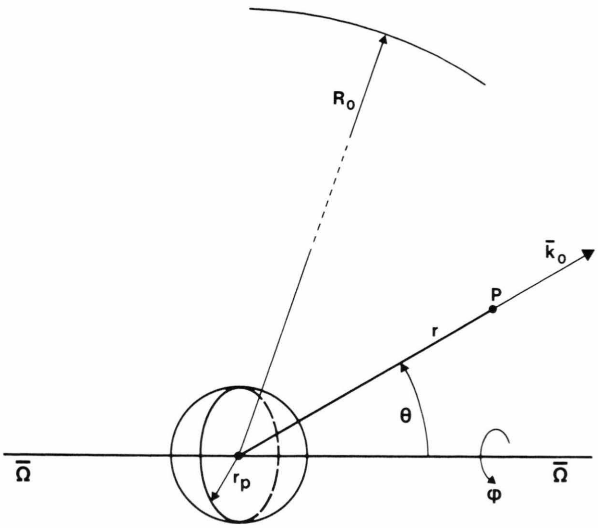

Fig. 1. Coordinate system showing the vector $\boldsymbol{k}_0$ and the hypothetical radial boundaries r_p and R_0.

If we now impose the boundary condition that

$$\alpha \ln(R_0/r_p) \equiv 1\,, \tag{11}$$

(9) and (10) then exactly yield the Dirac results for the relativistic electron [5].

The boundary condition (11) can actually be arrived at on the basis of a rigorous mathematical criterion. In obtaining the monopole solution (7) by integrating (1), we have in fact introduced a constant of integration ± 1, which yields the stated θ-dependence for the two spin directions. We are free, however, to renormalize this choice of integration constant and write instead for the θ-dependence

$$\left\{\cos\theta \pm \frac{1}{\alpha \ln(R_0/r_p)}\right\}/\sin\theta\,. \tag{12}$$

With this, the foregoing calculations yield exactly $\pm(m_0 c)\sin 2\omega_0 t$ for the linear momentum of the monopole, and $\mp \hbar/2$ for its angular momentum component [5].

If we now impose as a boundary condition that the monopole fields be nonsingular along the spin direction ($\theta = 0$ or π), we then arrive at the condition (11) automatically from (12). In this fashion, the relation (11) and the ensuing charge quantization follow rigorously from a mathematical constraint of regularity.

Regularizing logarithmic relations of this type between fundamental constants is certainly not new [8, 9]. However, it is perhaps the first time that such a relation emerges as a consequence of a monopole field structure as given in (7).

Finally, we note that, with $\alpha \simeq 1/137$, (11) yields the very large number $R_0/r_p = \exp(1/\alpha) \simeq 10^{60} = 10^{27}\,\text{cm}/10^{-33}\,\text{cm}$, of the same order of magnitude as the ratio of the so-called universe radius to the Planck length.

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